Synthesis of Quantized Feedback Control Software for Discrete Time Linear Hybrid Systems

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Abstract. We present an algorithm that given a *Discrete Time Linear Hybrid System* $\mathcal{H}$ returns a correct-by-construction software implementation K for a (*near time optimal*) robust quantized feedback controller for $\mathcal{H}$ along with the set of states on which K is guaranteed to work correctly (*controllable region*). Furthermore, K has a *Worst Case Execution Time* linear in the number of bits of the quantization schema.

1 Introduction

Software generation from models and formal specifications forms the core of model based design of embedded software [18]. This approach is particularly interesting for control systems since in such a case system specifications are much easier to define than the control software behavior itself.

A *control system* consists of two subsystems (forming the *closed loop* system): the *controller* and the *plant*. In an endless loop the controller measures outputs from and sends commands to the plant in order to drive it towards a given *goal*. In our setting the controller consists of software implementing the *control law*. System requirements are typically given as specifications for the closed loop system. Control engineering techniques are used to design the *control law* (i.e. the functional specifications for the control software) from the closed loop system specifications. Software engineering techniques are then used to design *control software* implementing a given control law.

Unfortunately, when the plant model is a *hybrid system* [4, 1, 3] existence of a control law is undecidable (e.g. see [17]) even for linear hybrid automata. This scenario is further complicated by the *quantization* process always present in software based control systems. Namely, measures from *sensors* go through an AD (analog-to-digital) conversion before being sent to the control software and commands from the control software go through a DA (digital-to-analog) conversion before being sent to plant *actuators*. Furthermore, typically a *robust* control is desired, that is, one that meets the given closed loop requirements notwithstanding (nondeterministic) variations in the plant parameters.

As for hybrid systems, no approach is available for the automatic synthesis of robust quantized feedback control laws *and* of their software implementation. This motivates the focus of our paper.

Our Main Contributions A *Discrete Time Linear Hybrid System* (DTLHS) is a discrete time hybrid system whose dynamics is defined as the logical conjunction of linear constraints on its continuous as well as discrete variables.

We present an effective algorithm that, given a DTLHS model $\mathcal{H}$ for the plant and a quantization schema (i.e. how many bits we use for AD conversion), returns a pair (K, R), where: K is a correct-by-construction software implementation (C language in our case) of a (*near time optimal*) *Quantized Feedback Controller* (QFC) for $\mathcal{H}$ and R is an OBDD [10] representation of the set of states (*controllable region*) on which K is guaranteed to meet the closed loop requirements. Furthermore, K is *robust* with respect to nondeterministic variations in the plant parameters and has a *Worst Case Execution Time* (WCET) guaranteed to be linear in the number of bits of the quantization schema.

We implemented our algorithm on top of the CUDD package and of the GLPK *Mixed Integer Linear Programming* (MILP) solver and present experimental results on using our tool to synthesize robust QFCs for a widely used mixed-mode analog circuit: the buck DC-DC converter (e.g. see [26]).

Analog DC-DC converters are a vital part of many mission (e.g. satellites) or safety (e.g. aircrafts) critical applications. However the ever increasing demand for energy efficiency and easy reconfigurability makes fully software based switching converters (e.g., as in [26]) a very attractive alternative to analog ones. Unfortunately, lack of formal reliability assessment (in order to bound the failure probability to 10^{-9}) limits the deployment of software based switching converters in safety critical applications. Reliability analysis for switching converters using an analog control schema has been studied in [13]. For software based converters, carrying out such a reliability analysis entails formal verification of the control law as well as of its software implementation. The above considerations make the buck DC-DC converter a very interesting (and challenging) example for automatic synthesis of correct-by-construction control software.

Our experimental results show that within about 20 hours of CPU time and within 200MB of RAM we can synthesize (K, R) as above for a 10 bit quantized buck DC-DC converter.

Related Work Synthesis of *Quantized Feedback Control Laws* for linear systems has been widely studied in control engineering (e.g. see [14]). However, to the best of our knowledge, no previously published paper addresses synthesis of *Quantized Feedback Control Software* for DTLHSs. Indeed, our work differs from previously published ones in the following aspects: (1) we provide a *tool* for automatic synthesis of correct-by-construction control software (rather than *design methodologies* for the control law); (2) we synthesize *robust control software* (thus encompassing quantization) whereas robust control law design techniques do not take into account the software implementation; (3) in order to generate provably correct software, we assume a nondeterministic (*malicious*) model for quantization errors rather than a stochastic one, as usually done in control engineering; (4) our synthesis tool also returns the *controllable region*, that is the set of states on which the synthesized control software is guaranteed to work correctly (this is very important for *Fault Detection Isolation and Recovery*, FDIR, e.g. see [21]). In the following we discuss some related literature.

Quantization can be seen as a form of abstraction, where the abstract state space and transitions are defined by the number of bits of AD conversion. Abstraction for hybrid systems has been widely studied. For example, see [25, 2, 20, 19] and citations thereof. Note however that all published literature on abstraction focuses on designing abstractions to support verification or control law design. In our case instead, the abstraction is fully defined by the AD conversion schema and our focus is on devising techniques to effectively remove abstract transitions in order to counteract the nondeterminism (information loss) stemming from the quantization process.

Control synthesis for *Timed Automata* (TA) [4], *Linear Hybrid Automata* (LHA) [1, 3] as well as nonlinear hybrid systems has been extensively studied. Examples are in [22, 11, 6, 30, 16, 28, 5, 9, 8] and citations thereof. We note however that all above papers address design of control laws and do not take into account the quantization process, that is, they assume *exact* (i.e. real valued) state measures. Here instead we address design of quantized feedback control software.

Correct-by-construction software synthesis in a finite state context has been studied in [7, 29, 27, 12]. The above approaches cannot be directly used in our context since they do not account for continuous state variables.

2 Background

Unless otherwise stated each variable x ranges on a known bounded interval $\mathcal{D}_x$ either of the reals or of the integers (discrete variables). We denote with $\sup(x)$ ($\inf(x)$) the sup (inf) of $\mathcal{D}_x$. Boolean variables are discrete variables ranging on the set $\mathbb{B} = \{0, 1\}$. We denote with $X = [x_1, \dots x_n]$ a finite sequence (list) of variables, with $\cup$ list concatenation and with $\mathcal{D}_X = \prod_{x \in X} \mathcal{D}_x$ the domain of X.

A *valuation* $X^* \in \mathcal{D}_X$ over a list of variables X is a function v that maps each variable $x \in X$ to a value $v(x)$ in $\mathcal{D}_x$. We may use the same notation to denote a variable (a syntactic object) and one of its valuations. The intended meaning will be always clear from the context. To clarify that a variable [valuation] x is real (integer, boolean) valued we may write x^r (x^d, x^b). Analogously X^r (X^d, X^b) denotes the sequence of real (integer, boolean) variables [valuations] in X. If x is a boolean variable [valuation] we write $\bar{x}$ for $(1 - x)$.

A *linear expression* (over X) is a linear combination with real coefficients of variables in X. A *constraint* (over X) is an expression of the form $\alpha \bowtie b$ where α is a linear expression over X, $\bowtie$ is one of $\leq$, $\geq$, $=$ and b is a real constant. A constraint is a *predicate* on X. If $A(X)$ and $B(X)$ are *predicates* on X, then $(A(X) \wedge B(X))$ and $(A(X) \vee B(X))$ are *predicates* on X. A *conjunctive predicate* is just a conjunction of linear constraints. A *satisfying assignment* to $P(X)$ is a valuation X^* such that $P(X^*) = 1$. Abusing notation we may denote with P the set of satisfying assignments to $P(X)$. Given a predicate $P(X)$ and a fresh boolean variable $y \notin X$, the *if-then predicate* $y \rightarrow P(X)$ $[\bar{y} \rightarrow P(X)]$ denotes the predicate $((y = 0) \vee P(X))$ $[((y = 1) \vee P(X))]$. In our setting (bounded variables), for any predicate $P(X)$ there exists a sequence Z of fresh boolean variables and a conjunctive predicate $Q(Z, X)$ s.t. $\forall X[P(X) \iff \exists Z\ Q(Z, X)]$ (see [23] for details). Thus, any if-then predicate can be transformed into a conjunctive predicate. Accordingly, we will regard and use if-then predicates as conjunctive predicates.

A *Mixed Integer Linear Programming* (MILP) problem with *decision variables* X is a tuple (max, $J(X)$, $A(X)$) where: X is a list of variables, $J(X)$ (*objective function*) is a linear expression on X and $A(X)$ (*constraints*) is a conjunctive predicate on X. A solution to $(\max, J(X), A(X))$ is a valuation X^* s.t. $A(X^*)$ holds and, for any valuation Ξ, $(A(\Xi) \rightarrow (J(\Xi) \leq J(X^*)))$. We write $(\min, J(X), A(X))$ for $(\max, -J(X), A(X))$. A *feasibility* problem is a MILP problem of the form $(\max, 0, A(X))$. We write also $A(X)$ for $(\max, 0, A(X))$.

A *Labeled Transition System* (LTS) is a tuple $\mathcal{S} = (S, A, T)$ where: S is a (possibly infinite) set of states, A is a (possibly infinite) set of *actions*, $T : S \times A \times S \rightarrow \mathbb{B}$ is the *transition relation* of $\mathcal{S}$. Let $s \in S$ and $a \in A$. We denote with: $\mathrm{Adm}(\mathcal{S}, s)$ the set of actions admissible in s, that is $\mathrm{Adm}(\mathcal{S}, s) = \{a \in A \mid \exists s' T(s, a, s')\}$ and with $\mathrm{Img}(\mathcal{S}, s, a)$ the set of next states from s via a, that is $\mathrm{Img}(\mathcal{S}, s, a) = \{s' \in S \mid T(s, a, s')\}$. A *run* or *path* for $\mathcal{S}$ is a sequence $\pi = s(0)a(0)s(1)a(1)s(2)a(2)\ldots$ of states $s(t)$ and actions $a(t)$ such that $\forall t \geq 0$ $T(s(t), a(t), s(t+1))$. The length $|\pi|$ of a run π is the number of actions in π. We denote with $\pi^{(S)}(t)$ the t-th state element of π, and with $\pi^{(A)}(t)$ the t-th action element of π. That is $\pi^{(S)}(t) = s(t)$, and $\pi^{(A)}(t) = a(t)$.

3 Discrete Time Linear Hybrid Systems

In this section we introduce *Discrete Time Linear Hybrid Systems* (DTLHS).

Definition 1. *A* Discrete Time Linear Hybrid System *(DTLHS) is a tuple* $\mathcal{H} = (X, U, Y, N)$ *where:*

- $X = X^r \cup X^d$ *is a finite sequence of real* (X^r) *and discrete* (X^d) present state *variables. We denote with* X' *the sequence of* next state *variables obtained by decorating with* $'$ *all variables in* X.
- $U = U^r \cup U^d$ *is a finite sequence of* input *variables.*
- $Y = Y^r \cup Y^d$ *is a finite sequence of* auxiliary *variables. Auxiliary variables are typically used to model* modes *(e.g., from switching elements such as diodes) or* uncontrollable inputs *(e.g., disturbances).*
- $N(X, U, Y, X')$ *is a* conjunctive predicate *over* $X \cup U \cup Y \cup X'$ *defining the transition relation (*next state*) of the system.*

Note that in our setting (bounded variables) any predicate can be transformed into a conjunctive predicate (Sect. 2). Accordingly, in Def. 1, without loss of generality we focused on conjunctive predicates in order to simplify our exposition.

The dynamics of a DTLHS $\mathcal{H} = (X, U, Y, N)$ is defined by $\mathrm{LTS}(\mathcal{H}) = (\mathcal{D}_X, \mathcal{D}_U, \bar{N})$ where: $\bar{N} : \mathcal{D}_X \times \mathcal{D}_U \times \mathcal{D}_X \rightarrow \mathbb{B}$ is a function s.t. $\bar{N}(s, a, s') = \exists\, y \in \mathcal{D}_Y\ N(s, a, y, s')$. A *state* for $\mathcal{H}$ is a state for $\mathrm{LTS}(\mathcal{H})$ and a *run* (or *path*) for $\mathcal{H}$ is a run for $\mathrm{LTS}(\mathcal{H})$ (Sect. 2).

Example 1. Let $\mathcal{H} = (\{x\}, \{u\}, \varnothing, N)$ with $\mathcal{D}_x = [-2.5, 2.5]$, $\mathcal{D}_u = \{0, 1\}$, and $N(x, u, x') = [\overline{u} \rightarrow x' = \alpha x] \wedge [u \rightarrow x' = \beta x]$ with $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$. When $Y = \varnothing$ (as here) for the sake of simplicity we omit it from N arguments.

Adding nondeterminism to $\mathcal{H}$ allows us to synthesize *robust* controllers. For example, variations in the parameter α can be modelled with a tolerance $\rho \in [0, 1]$ (e.g., $\rho = 0.5$) for α. This replaces N with: $N^\rho = [\overline{u} \rightarrow x' \leq (1 + \rho)\alpha x] \wedge [\overline{u} \rightarrow x' \geq (1 - \rho)\alpha x] \wedge [u \rightarrow x' = \beta x]$. Suitable control synthesis on $\mathcal{H}^\rho = (\{x\}, \{u\}, \varnothing, N^\rho)$ will yield a *robust* (up to ρ) controller for $\mathcal{H}$.

Example 2. The buck DC-DC converter (right part of Fig. 1) is a mixed-mode analog circuit converting the DC input voltage (V_i in Fig. 1) to a desired DC output voltage (v_O in Fig. 1). The typical software based approach (e.g. see [26]) is to control the switch u in Fig. 1 (typically implemented with a MOSFET) with a microcontroller. Designing the software to run on the microcontroller to properly actuate the switch is the control design problem for the buck DC-DC converter in our context. The circuit in Fig. 1 can be modeled as a DTLHS $\mathcal{H} = (X, U, Y, N)$ with: $X = X^r = [i_L, v_O]$, $U = U^d = [u]$, $Y = Y^r \cup Y^d$ with $Y^r = [i_u, v_u, i_D, v_D]$ and $Y^d = [q]$. $\mathcal{H}$ auxiliary variables Y stem from the constitutive equations of the switching elements (i.e. the switch u and the diode D in Fig. 1). The transition relation $N(X, U, Y, X')$ for $\mathcal{H}$ is shown in Fig. 1 (left) where we use a discrete time model with sampling time T (writing x' for $x(t+1)$) and model a tolerance $\rho = 0.25$ (25%) on V_i values. In Fig. 1 (left), constants $a_{i,j}, b_{i,j}$ depend on the circuit parameters R, r_L, r_C, L, C and algebraic constraints (i.e. constraints not involving next state variables) stem from the constitutive equations of the switching elements (see [23] for details).

$$
\begin{aligned}
&N(X,U,Y,X') = ((i_L{}' = (1+Ta_{1,1})i_L + Ta_{1,2}v_O + Tb_{1,1}v_D) \\
&\wedge\ (v_O{}' = Ta_{2,1}i_L + (1+Ta_{2,2})v_O + Tb_{2,1}v_D) \\
&\wedge\ (v_u - v_D \leq (1+\rho)V_i) \wedge (v_u - v_D \geq (1-\rho)V_i) \\
&\wedge\ (i_D = i_L - i_u) \wedge (q \rightarrow v_D = 0) \wedge (q \rightarrow i_D \geq 0) \\
&\wedge\ (\bar{q} \rightarrow v_D \leq 0) \wedge (\bar{q} \rightarrow v_D = R_{off} i_D) \\
&\wedge\ (u \rightarrow v_u = 0) \wedge (\bar{u} \rightarrow v_u = R_{off} i_u))
\end{aligned}
$$

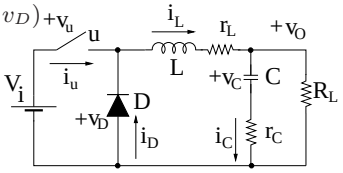


Fig. 1. Buck DC-DC converter

4 Quantized Feedback Control Problem for DTLHS

We define the *Feedback Control Problem* for LTSs (Def. 2) and for DTLHSs (Def. 4). On such a base we define the *Quantized Feedback Control Problem* for DTLHSs (Def. 6).

We begin by extending to (possibly infinite) LTSs the definitions in [29, 12] for finite LTSs. In what follows, let $\mathcal{S} = (S, A, T)$ be an LTS, I, $G \subseteq S$ be, respectively, the *initial* and *goal* sets of $\mathcal{S}$.

Definition 2. *A* controller *for $\mathcal{S}$ is a function $K : S \times A \rightarrow \mathbb{B}$ s.t. $\forall s \in S$, $\forall a \in A$, if $K(s,a)$ then $\exists s'\ T(s,a,s')$. $\mathrm{Dom}(K)$ denotes the set of states for which at least a control action is enabled. Formally,* $\mathrm{Dom}(K) = \{s \in S \mid \exists a\ K(s,a)\}$. *$\mathcal{S}^{(K)}$ denotes the* closed loop system*, that is the LTS $(S, A, T^{(K)})$, where $T^{(K)}(s,a,s') = T(s,a,s') \wedge K(s,a)$. A* control problem *for $\mathcal{S}$ is a triple $(\mathcal{S}, I, G)$.*

A *controller* for $\mathcal{S}$ (Def. 2) is used to restrict $\mathcal{S}$ behavior so that all states in the initial region (I) will reach in one or more steps the goal region (G). In the following, we formalize such a concept by defining strong and weak solutions to an LTS control problem.

We call a path π *fullpath* [7] if either it is infinite or its last state $\pi^{(S)}(|\pi|)$ has no successors. We denote with $\mathrm{Path}(s)$ the set of fullpaths starting in state s, i.e. the set of fullpaths π such that $\pi^{(S)}(0) = s$. Observe that $\mathrm{Path}(s)$ is never empty, since it contains at least the path of length 0 containing only state s.

Given a path π in $\mathcal{S}$, $J(\mathcal{S}, \pi, G)$ denotes the unique $n > 0$, if it exists, s.t. $[\pi^{(S)}(n) \in G] \wedge [\forall\ 0 < i < n . \pi^{(S)}(i) \notin G]$, $+\infty$ otherwise. We require $n > 0$ since our systems are nonterminating and each controllable state (including a goal

state) must have a path of positive length to a goal state. The *worst case distance* (*pessimistic view*) of a state s from the goal region G is $J_{strong}(\mathcal{S}, G, s) = \sup\{J(\mathcal{S}, \pi, G) \mid \pi \in \mathrm{Path}(s)\}$. The *best case distance* (*optimistic view*) of a state s from the goal region G is $J_{weak}(\mathcal{S}, G, s) = \inf\{J(\mathcal{S}, \pi, G) \mid \pi \in \mathrm{Path}(s)\}$.

Definition 3. *A* strong [weak] *solution to a control problem $\mathcal{P} = (\mathcal{S}, I, G)$ is a controller K for $\mathcal{S}$, such that $I \subseteq \mathrm{Dom}(K)$ and for all $s \in \mathrm{Dom}(K)$, $J_{strong}(\mathcal{S}^{(K)}, G, s)$ $[J_{weak}(\mathcal{S}^{(K)}, G, s)]$ is finite.*

Example 3. Let $\mathcal{S}_0$ $[\mathcal{S}_1]$ be the LTS which transition relation consists of the continuous [all] arrows in Fig. 2 (left). Let $\hat{I} = \{-1, 0, 1\}$ and $\hat{G} = \{0\}$. Then, $\hat{K}(\hat{s}, \hat{u}) \equiv [\hat{s} \neq 0 \Rightarrow \hat{u} = 0]$ is a strong solution to the control problem $(\mathcal{S}_0, \hat{I}, \hat{G})$ and a weak solution to $(\mathcal{S}_1, \hat{I}, \hat{G})$.

Remark 1. Note that if K is a strong solution to $(\mathcal{S}, I, G)$ and $G \subseteq I$ (as it is usually the case in control problems) then all paths starting from $\mathrm{Dom}(K)$ $(\subseteq I)$ will *touch* G infinitely often (*stability*).

A DTLHS control problem is a triple $(\mathcal{H}, I, G)$ where $\mathcal{H}$ is a DTLHS and $(\mathrm{LTS}(\mathcal{H}), I, G)$ is an LTS control problem. For DTLHSs we restrict ourselves to control problems where I and G can be represented as conjunctive predicates. From [17] it is easy to show that DTLHS control problems are undecidable [23]. For DTLHS control problems usually *robust* controllers are desired. That is, controllers that, notwithstanding nondeterminism in the plant (e.g. due to parameter variations), drive the plant state to the goal region. For this reason, and to counteract the nondeterminism stemming from the quantization process, we focus on strong solutions. Furthermore, to accommodate quantization errors, always present in software based controllers, it is useful to relax the notion of control solution by tolerating an (arbitrarily small) error ε on the continuous variables. This leads to the definition of ε-solution. Let ε be a nonnegative real number, $W^r = \prod_{i=1}^{n} W_i^r \subseteq \mathcal{D}_X^r$, $W^d = \prod_{i=1}^{m} W_i^d \subseteq \mathcal{D}_X^d$ and $W = W^r \times W^d \subseteq \mathcal{D}_X^r \times \mathcal{D}_X^d$. The *$\varepsilon$-relaxation* of W is the set (*ball* of radius ε) $\mathcal{B}_\varepsilon(W) = \{(z_1, \ldots z_n, q_1, \ldots q_m) \mid (q_1, \ldots q_m) \in W^d$ and $\forall i \in \{1, \ldots n\}\ \exists\, x_i \in W_i^r$ s.t. $|z_i - x_i| \leq \varepsilon\}$.

Definition 4. *Let $(\mathcal{H}, I, G)$ be a DTLHS control problem and ε be a nonnegative real number. An ε-solution to $(\mathcal{H}, I, G)$ is a* strong solution *to the LTS control problem* $(\mathrm{LTS}(\mathcal{H}), I, \mathcal{B}_\varepsilon(G))$.

Example 4. Let $\mathcal{P} = (\mathcal{H}, I, G)$, $\mathcal{H}$ as in Ex. 1, $I = \mathcal{D}_x$ and $G = \{0\}$ (represented by conjunctive predicate $x = 0$). Control problem $\mathcal{P}$ has no solution (because of the Zeno phenomenon), but for all $\varepsilon > 0$ it has the ε-solution K s.t. $\forall x \in I.\ K(x, 0) = 1$.

Example 5. The typical goal of a controller for the buck DC-DC converter in Ex. 2 is keeping the output voltage v_O *close enough* to a given reference value V_{ref}. This leads to the control problem $\mathcal{P} = (\mathcal{H}, I, G)$ where: $\mathcal{H}$ is defined in Ex. 2, $I = (|i_L| \leq 2) \wedge (0 \leq v_O \leq 6.5)$, $G = (|v_O - V_{ref}| \leq \theta) \wedge (|i_L| \leq 2)$ and $\theta = 0.01$ is the desired converter precision.

In order to define quantized feedback control problems for DTLHSs (Def. 6) we introduce *quantizations* (Def. 5). Let x be a real valued variable ranging on a bounded interval of reals $\mathcal{D}_x = [a_x, b_x]$. A *quantization* for x is a function γ

from $\mathcal{D}_x$ to a bounded interval of integers $\gamma(\mathcal{D}_x) = [\hat{a_x}, \hat{b_x}]$. For ease of notation we extend quantizations to integer variables ranging on a bounded interval of integers by stipulating that the only quantization γ for such variables is the identity function (i.e. $\gamma(x) = x$). The *width* $\left\|\gamma^{-1}(v)\right\|$ of $v \in \gamma(\mathcal{D}_x)$ in γ is defined as follows: $\left\|\gamma^{-1}(v)\right\| = \sup\{\ |w - z| \mid w, z \in \mathcal{D}_x\ \wedge\ \gamma(w) = \gamma(z) = v\}$. The *quantization step* $\|\gamma\|$ is defined as follows: $\|\gamma\| = \max\left\{\ \left\|\gamma^{-1}(v)\right\|\ \mid v \in \gamma(\mathcal{D}_x)\right\}$.

Definition 5. *Let $\mathcal{H} = (X, U, Y, N)$ be a DTLHS. A* quantization *Γ for $\mathcal{H}$ is a set of maps $\Gamma = \{\gamma_w \mid \gamma_w$ is a quantization for $w \in X \cup U\}$. Let $W = [w_1, \ldots w_k] \subseteq X \cup U$ and $v = [v_1, \ldots v_k] \in \mathcal{D}_W$. We write $\Gamma(v)$ for the tuple $[\gamma_{w_1}(v_1), \ldots \gamma_{w_k}(v_k)]$ and $\Gamma(\mathcal{D}_W)$ for the set of tuples $\{\Gamma(v) \mid v \in \mathcal{D}_W\}$. Finally, the* quantization step *$\|\Gamma\|$ for Γ is defined as: $\|\Gamma\| = \max\{\ \|\gamma\| \mid \gamma \in \Gamma\}$.*

A control problem admits a *quantized* solution if control decisions can be made by just looking at quantized values. This enables a software implementation for a controller.

Definition 6. *Let $\mathcal{H} = (X, U, Y, N)$ be a DTLHS, Γ be a quantization for $\mathcal{H}$ and $\mathcal{P} = (\mathcal{H}, I, G)$ be a control problem. A Γ* Quantized Feedback Control *(QFC) solution to $\mathcal{P}$ is a $\|\Gamma\|$-solution $K(x, u)$ to $\mathcal{P}$ such that $K(x, u) = \hat{K}(\Gamma(x), \Gamma(u))$ where $\hat{K} : \Gamma(\mathcal{D}_X) \times \Gamma(\mathcal{D}_U) \to \mathbb{B}$.*

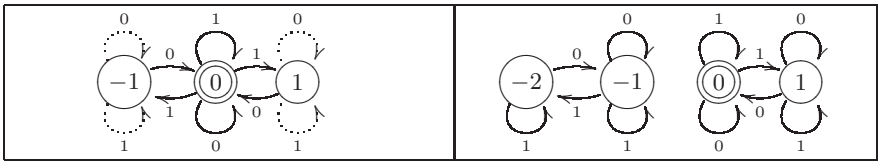


Fig. 2. Γ control abstraction for DTLHSs in Exs. 3 and 8 (left) and Ex. 9 (right)

Example 6. Let $\mathcal{P}$ be as in Ex. 4, $\Gamma(x) = round(x/2)$ (where $round(x) = \lfloor x \rfloor + \lfloor 2(x - \lfloor x \rfloor) \rfloor$ is the usual rounding function) and $\hat{K}$ as in Ex. 3. Then, $\|\Gamma\| = 2$ and $K(x, u) = \hat{K}(\Gamma(x), \Gamma(u))$ is a Γ QFC solution to $\mathcal{P}$.

5 Control Abstraction

The AD process maps intervals of state values into discrete state values. As a result the control software *sees* the controlled system (plant) as a nondeterministic finite automaton. Of course we want our control software to work notwithstanding such a nondeterminism (strong solution, Def. 3). To this end we should try to *limit* such a nondeterminism as much as possible. This leads to the notion of *control abstraction* (Def. 8), the main focus of this section.

Since QFC (Def. 6) rests on AD conversion we must be careful not to drive the plant outside the bounds in which AD conversion works correctly. This leads to the definition of *safe action* (Def. 7). Intuitively, an action is safe in a state if it never drives the system outside of its state bounds.

Definition 7. *Let $\mathcal{H} = (X, U, Y, N)$ be a DTLHS and Γ be a quantization.*

1. *We say that action $u \in \mathcal{D}_U$ is* safe *for $s \in \mathcal{D}_X$ (in $\mathcal{H}$) if for all s', $[\exists y \in \mathcal{D}_Y\ N(s, u, y, s')$ implies $s' \in \mathcal{D}_X]$.*

2. *We say that action $\hat{u} \in \Gamma(\mathcal{D}_U)$ is Γ-safe in state $\hat{s} \in \Gamma(\mathcal{D}_X)$ if for all $s \in \Gamma^{-1}(\hat{s})$, $u \in \Gamma^{-1}(\hat{u})$, u is safe for s in $\mathcal{H}$.*

Note that, in general, not all actions $u \in \mathcal{D}_U$ are safe in $\mathcal{H}$ since Def. 1 only asks N to be a conjunctive predicate.

Example 7. Let $\mathcal{H}$ be as in Ex. 1. Then action $u = 1$ is not safe in state $s = 2$ since we have $N(2, 1, 3)$, and $s' = 3$ is outside $\mathcal{H}$ state bounds.

A *control abstraction* (Def. 8) is a finite state automaton modelling how a DTLHS is *seen* from the control software because of AD conversion.

Definition 8. *Let $\mathcal{H} = (X, U, Y, N)$ be a DTLHS and Γ be a quantization for $\mathcal{H}$. We say that the LTS $\hat{\mathcal{H}} = (\Gamma(\mathcal{D}_X), \Gamma(\mathcal{D}_U), \hat{N})$ is a Γ* control abstraction *of $\mathcal{H}$ if its transition relation $\hat{N}$ satisfies the following conditions.*

1. *Each abstract transition stems from a concrete transition. Formally: for all $\hat{s}, \hat{s}' \in \Gamma(\mathcal{D}_X)$, $\hat{u} \in \Gamma(\mathcal{D}_U)$, if $\hat{N}(\hat{s}, \hat{u}, \hat{s}')$ then there exist $s \in \Gamma^{-1}(\hat{s})$, $u \in \Gamma^{-1}(\hat{u})$, $s' \in \Gamma^{-1}(\hat{s}')$, $y \in \mathcal{D}_Y$ s.t. $N(s, u, y, s')$.*
2. *If an abstract action is safe then all its possible concrete effects (besides self-loops) are faithfully represented in the abstract system. Formally: for all $\hat{s} \in \Gamma(\mathcal{D}_X)$, $\hat{u}$ Γ-safe in $\hat{s}$, $s \in \Gamma^{-1}(\hat{s})$, $u \in \Gamma^{-1}(\hat{u})$, $s' \in \mathcal{D}_X$, if $[\exists y \in \mathcal{D}_Y\ N(s, u, y, s')]$ and $\Gamma(s) \neq \Gamma(s')$ then $\hat{N}(\Gamma(s), \Gamma(u), \Gamma(s'))$.*
3. *If there is no upper bound to the length of concrete paths inside the counter-image of an abstract state then there is an (abstract) self-loop. Formally: for all $\hat{s} \in \Gamma(\mathcal{D}_X)$, $\hat{u} \in \Gamma(\mathcal{D}_U)$, if $\forall k\ \exists x(0), \ldots x(k+1) \in \Gamma^{-1}(\hat{s})$ $\exists u(0), \ldots u(k) \in \Gamma^{-1}(\hat{u})\ \exists y(0), \ldots y(k) \in \mathcal{D}_Y\ [\bigwedge_{t=0}^{k} N(x(t), u(t), y(t), x(t+1))]$ then $\hat{N}(\hat{s}, \hat{u}, \hat{s})$.*

We say that $\hat{\mathcal{H}}$ is a control abstraction *of $\mathcal{H}$ if $\hat{\mathcal{H}}$ is a Γ* control abstraction *of $\mathcal{H}$ for some quantization Γ. Finally, we denote with $\mathcal{A}_\Gamma(\mathcal{H})$ the set of all Γ control abstractions on $\mathcal{H}$.*

Note that any abstraction (e.g. see [2]) is also a control abstraction. However, the converse is false since some concrete transition (e.g. a self loop or an unsafe action) may have no abstract image. Let $\mathcal{S}_1 = (S, A, T_1)$ and $\mathcal{S}_2 = (S, A, T_2)$ be LTSs. We say that $\mathcal{S}_1$ *refines* $\mathcal{S}_2$ (notation $\mathcal{S}_1 \sqsubseteq \mathcal{S}_2$) iff for each $s, s' \in S$, $a \in A$, $T_1(s, a, s')$ implies $T_2(s, a, s')$. The binary relation $\sqsubseteq$ is a partial order. Moreover, the poset $(\mathcal{A}_\Gamma(\mathcal{H}), \sqsubseteq)$ is a *lattice*. Furthermore, since $\mathcal{A}_\Gamma(\mathcal{H})$ is a finite set, the poset $(\mathcal{A}_\Gamma(\mathcal{H}), \sqsubseteq)$ has unique *maximum* and unique *minimum* elements.

Example 8. Let $\mathcal{H}$ be as in Ex. 1 and Γ be as in Ex. 6. Each Γ control abstraction of $\mathcal{H}$ has the form $\hat{\mathcal{H}} = (\{-1, 0, 1\}, \{0, 1\}, \hat{N})$, where the set of transitions in $\hat{N}$ is any subset, containing all continuous arrows, of the set of transitions of the automaton depicted in Fig. 2 (left). In particular, a control abstraction may omit some self loops (namely, those with dotted arrows in Fig. 2). Transitions $\hat{N}(0, 0, 0)$ and $\hat{N}(0, 1, 0)$ must belong to all Γ control abstractions, because of condition 3 in Def. 8. In fact all paths starting in 0 will remain in 0 forever. The transition relation defined in Fig. 2 (left) by continuous arrows is the minimum Γ control abstraction $\hat{\mathcal{H}}_{min}$ of $\mathcal{H}$ whereas the transition relation defined by all arrows is the maximum Γ control abstraction $\hat{\mathcal{H}}_{max}$ of $\mathcal{H}$. Note that there is no controller (strongly) driving all states of $\hat{\mathcal{H}}_{max}$ to state 0. In fact, because of self-loops, action 0 from state 1 may lead to state 0 as well as to state 1 (self-loop). On the other hand the controller $\hat{K}$ enabling only action 0 in any

state will (weakly) drive all states of $\hat{\mathcal{H}}_{max}$ to 0 since each state in $\hat{\mathcal{H}}_{max}$ has at least a 0-labelled transition leading to state 0. Controller $\hat{K}$ will also (strongly and thus weakly) drive all states of $\hat{\mathcal{H}}_{min}$ (including 0) to state 0.

Remark 2. Example 8 suggests that we should focus on minimum control abstractions in order to increase our chances of finding a strong controller. Correctness of such an intuition will be shown in Theor. 1. As for computing the minimum control abstraction we note that this entails deciding if a given self-loop can be eliminated according to condition 3 in Def. 8. Unfortunately it is easy to show that such a problem comes down to solve a reachability problem on linear hybrid systems, that, by [17], is undecidable. Thus, self-loop eliminability is undecidable too in our context. As a result, in general, we cannot hope to compute *exactly* the minimum control abstraction.

6 Synthesis of Quantized Feedback Control Software

We outline our synthesis algorithm QKS (*Quantized feedback Kontrol Synthesis*) and give its properties (Theor. 1). Details are in [23]. QKS takes as input a tuple $(\Gamma, \mathcal{H}, I, G)$, where: $\mathcal{H} = (X, U, Y, N)$ is a DTLHS, Γ is a quantization for $\mathcal{H}$ and $(\mathcal{H}, I, G)$ is a control problem. QKS returns a tuple $(\mu, \hat{D}, \hat{K})$, where: $\mu \in \{\textsc{Sol}, \textsc{NoSol}, \textsc{Unk}\}$, $K(x,u) = \hat{K}(\Gamma(x), \Gamma(u))$ is a Γ QFC solution for $\mathcal{H}$ (Def. 6), $\hat{D} = \mathrm{Dom}(\hat{K})$ and $D = \Gamma^{-1}(\hat{D}) = \mathrm{Dom}(K)$ is K controllable region.

We compute QKS output as follows. As a first step we compute a Γ control abstraction $\hat{\mathcal{Q}} = (\Gamma(\mathcal{D}_X), \Gamma(\mathcal{D}_U), \hat{N})$ of $\mathcal{H}$ as close as we can (see Remark 2) to the minimum one. Sect. 6.1 (function minCtrAbs in Alg. 1) outlines how $\hat{\mathcal{Q}}$ can be computed. Let $\hat{I} = \Gamma(I)$, $\hat{G} = \Gamma(G)$ and $\hat{K}$ be the *most general optimal* (mgo) strong solution to the (LTS) control problem $(\hat{\mathcal{Q}}, \varnothing, \hat{G})$. Intuitively, the mgo strong solution $\hat{K}$ to a control problem $(\hat{\mathcal{Q}}, \varnothing, \hat{G})$ is the *unique* strong solution that, disallowing as few actions as possible, drives as many states as possible to a state in $\hat{G}$ along a shortest path. We compute (the OBDD representation for) $\hat{K}$ by implementing a suitable variant of the algorithm in [12]. Finally, we define: $K(x,u) = \hat{K}(\Gamma(x), \Gamma(u))$, $\hat{D} = \mathrm{Dom}(\hat{K})$, and $D = \Gamma^{-1}(\hat{D}) = \mathrm{Dom}(K)$.

If $\hat{I} \subseteq \hat{D}$ then QKS returns $\mu = \textsc{Sol}$. Note that in such a case, from the construction in [12], $\hat{K}$ is time optimal for the control problem $(\hat{\mathcal{Q}}, \hat{I}, \hat{G})$, thus K will *typically* move along a shortest path to G (i.e., K is *near time-optimal*). If $\hat{I} \not\subseteq \hat{D}$ then we compute the maximum Γ control abstraction $\hat{\mathcal{W}}$ of $\mathcal{H}$ and use the algorithm in [29] to check if there exists a weak solution to $(\hat{\mathcal{W}}, \hat{I}, \hat{G})$. If that is the case QKS returns $\mu = \textsc{Unk}$, otherwise QKS returns $\mu = \textsc{NoSol}$. Note that the maximum control abstraction may contain also (possibly) unsafe transitions (condition 2 of Def. 8). Thus a weak solution for $\hat{\mathcal{W}}$ may exist even when no weak solution for $\hat{\mathcal{Q}}$ exists. Using the above notations Theor. 1 summarizes the main properties of QKS.

Theorem 1. *Let $\mathcal{H}$ be a DTLHS, Γ be a quantization and $(\mathcal{H}, I, G)$ be a control problem. Then* QKS*$(\Gamma, \mathcal{H}, I, G)$ returns a triple $(\mu, \hat{D}, \hat{K})$ s.t.: $\mu \in \{\textsc{Sol}, \textsc{NoSol}, \textsc{Unk}\}$, $\hat{D} = \mathrm{Dom}(\hat{K})$, $D = \Gamma^{-1}(\hat{D})$ and $K = \hat{K}(\Gamma(x), \Gamma(u))$ is a Γ QFC solution to the control problem $(\mathcal{H}, D, G)$. Furthermore, the following holds.*

1. *If* $\mu = \textsc{Sol}$ *then* $I \subseteq D$ *and* K *is a* Γ *QFC solution to the control problem* $(\mathcal{H}, I, G)$.
2. *If* $\mu = \textsc{NoSol}$ *then there is no* Γ *QFC solution to the control problem* $(\mathcal{H}, I, G)$.
3. *If* $\mu = \textsc{Unk}$ *then* QKS *is inconclusive, that is* $(\mathcal{H}, I, G)$ *may or may not have a* Γ *QFC solution.*

Note that the AD conversion hardware is modelled by Γ and that from the OBDD for $\hat{K}$ above we get a C program (Section 6.2). Thus $\hat{K}$ as described above defines indeed the control software we are looking for. Finally, note that case 3 in Theor. 1 stems from undecidability of the QFC problem [17].

Example 9. Let $\mathcal{P} = (\mathcal{H}, I, G)$ be as in Ex. 4 and Γ be as in Ex. 8. For all Γ control abstractions $\hat{\mathcal{H}}$ (and thus for the minimum one shown in Ex. 8) not containing the self loops $\hat{N}(-1, 0, -1)$ and $\hat{N}(1, 0, 1)$, $\hat{K}$ as in Ex. 3 is the mgo strong solution to $(\hat{\mathcal{H}}, \varnothing, \Gamma(G))$. Thus, $K(s, u)$ as in Ex. 6 is a Γ QFC solution to $\mathcal{P}$. Weak solutions to $(\hat{\mathcal{H}}, \Gamma(I), \Gamma(G))$ exist for all Γ control abstractions $\hat{\mathcal{H}}$. Note that existence of a Γ QFC solution to a control problem depends on Γ. Let us consider the quantization $\Gamma'(x)=\lfloor x/2 \rfloor$ for $\mathcal{H}$. Then the maximum Γ' control abstraction of $\mathcal{H}$ is $\mathcal{L} = (\{-2, -1, 0, 1\}, \{0, 1\}, \hat{N})$, where the transition $\hat{N}$ is depicted in Fig. 2 (right). Clearly $(\mathcal{L}, \Gamma'(I), \Gamma'(G))$ has no weak solution since there is no path to the goal $\Gamma'(G) = \{0\}$ from any of the states $-2, -1$. Thus $\mathcal{P}$ has no Γ' QFC solution.

6.1 Computing Control Abstractions

Function minCtrAbs in Alg. 1 computes a *close to minimum* Γ control abstraction (Def. 8) $\hat{\mathcal{Q}} = (\Gamma(\mathcal{D}_X), \Gamma(\mathcal{D}_U), \hat{N})$ of $\mathcal{H} = (X, U, Y, N)$ as well as $\hat{I} = \Gamma(I)$ and $\hat{G} = \Gamma(G)$.

Line 6 initializes (the OBDDs for) $\hat{N}$, $\hat{I}$, $\hat{G}$ to $\varnothing$ (i.e. the boolean function identically 0). Line 2 loops through all $|\Gamma(\mathcal{D}_X)|$ states $\hat{s}$ of $\hat{\mathcal{H}}$. Line 3 [line 4] add state $\hat{s}$ to $\hat{I}$ [$\hat{G}$] if $\hat{s}$ is the image of a concrete state in I [G]. Line 5 loops through all $|\Gamma(\mathcal{D}_U)|$ actions $\hat{u}$ of $\hat{\mathcal{H}}$. Line 13 checks if action $\hat{u}$ is Γ-safe in $\hat{s}$ (see Def. 7.2 and Def. 8.2). Function SelfLoop in line 7 returns 0 when, accordingly to Def. 8.3 a self-loop need not to be in $\hat{N}$. An exact check is undecidable (Remark 2), however our *gradient based* SelfLoop function typically turns out (Tab. 1 in Sect. 7) to be a quite tight overapproximation of the sets of (strictly needed) self-loops. We compute SelfLoop$(\hat{s}, \hat{u})$ as follows. For each real valued state component x_i, let $w_i[W_i]=$ (min[max], $x_i' - x_i, N(X, U, Y, X') \wedge \Gamma(X) = \hat{s} \wedge \Gamma(U) = \hat{u}$). If for some i [$w_i \neq 0 \wedge W_i \neq 0 \wedge$ (w_i and W_i have the same sign)] then SelfLoop returns 0 (since any *long enough* sequence of concrete actions in $\Gamma^{-1}(\hat{u})$ will drive state component x_i outside of $\Gamma^{-1}(\hat{s})$), otherwise SelfLoop returns 1. Lines 9, 10, 11 compute a *quite tight* overapproximation (Over_Img) of the set of states reachable in one step from $\hat{s}$. Line 12 loops on all |Over_Img| abstract next states $\hat{s}'$ that may be reachable with the abstract outgoing transition $(\hat{s}, \hat{u})$ under consideration. Line 13 checks if there exists a concrete transition realizing the abstract transition $(\hat{s}, \hat{u}, \hat{s}')$ when $\hat{s} \neq \hat{s}'$ (no self-loop) and if so adds the

abstract transition $(\hat{s}, \hat{u}, \hat{s}')$ to $\hat{N}$ (line 14). Finally, line 15 returns the (transition relation for) the control abstraction along with $\hat{I}$ and $\hat{G}$.

Remark 3. From the loops in lines 2, 5, 12 we see that the worst case runtime for Alg. 1 is $\mathcal{O}(|\Gamma(\mathcal{D}_X)|^2|\Gamma(\mathcal{D}_U)|)$. However, thanks to the heuristic in lines 9–11, Alg. 1 typical runtime is about $\mathcal{O}(|\Gamma(\mathcal{D}_X)||\Gamma(\mathcal{D}_U)|)$ as confirmed by our experimental results (Sect. 7, Fig. 3(b)).

Remark 4. Alg. 1 is explicit in the (abstract) states and actions of $\hat{\mathcal{H}}$ and symbolic with respect to the auxiliary variables (*modes*) in the transition relation N of $\mathcal{H}$. As a result our approach will work well with systems with just a few state variables and many modes, our target here.

Algorithm 1 Building control abstractions

Input: A quantization Γ, a DTLHS $\mathcal{H} = (X, U, Y, N)$, a control problem $(\mathcal{H}, I, G)$.
function minCtrAbs(Γ, $\mathcal{H}$, I, G):

1. $\hat{N} \leftarrow \varnothing, \hat{I} \leftarrow \varnothing, \hat{G} \leftarrow \varnothing$, **let** $X = [x_1, \ldots, x_n]$, $X' = [x'_1, \ldots, x'_n]$
2. **for all** $\hat{s} \in \Gamma(\mathcal{D}_X)$ **do**
3. **if** (MILP (min, 0, $I(X) \wedge \Gamma(X) = \hat{s}$) is feasible) **then** $\hat{I} \leftarrow \hat{I} \cup \{\hat{s}\}$
4. **if** (MILP (min, 0, $G(X) \wedge \Gamma(X) = \hat{s}$) is feasible) **then** $\hat{G} \leftarrow \hat{G} \cup \{\hat{s}\}$
5. **for all** $\hat{u} \in \Gamma(\mathcal{D}_U)$ **do**
6. **if** (MILP (min, 0, $N(X, U, Y, X') \wedge \Gamma(X) = \hat{s} \wedge \Gamma(U) = \hat{u} \wedge X' \notin \mathcal{D}_X$) is feasible) **then continue**
7. **if** SelfLoop($\hat{s}$, $\hat{u}$) **then** $\hat{N} \leftarrow \hat{N} \cup \{(\hat{s}, \hat{u}, \hat{s})\}$
8. **for all** $i = 1, \ldots n$ **do**
9. $m_i \leftarrow x'^*_i$, where $X'^* = [x'^*_1, \ldots, x'^*_n]$ is a solution to the MILP (min, x'_i, $N(X, U, Y, X') \wedge \Gamma(X) = \hat{s} \wedge \Gamma(U) = \hat{u}$)
10. $M_i \leftarrow x'^*_i$, where $X'^* = [x'^*_1, \ldots, x'^*_n]$ is a solution to the MILP (max, x'_i, $N(X, U, Y, X') \wedge \Gamma(X) = \hat{s} \wedge \Gamma(U) = \hat{u}$)
11. **let** Over_Img($\hat{s}, \hat{u}$) $= \prod_{i=1,\ldots n}[\gamma_{x_i}(m_i), \gamma_{x_i}(M_i)]$
12. **for all** $\hat{s}' \in$ Over_Img($\hat{s}, \hat{u}$) **do**
13. **if** $\hat{s} \neq \hat{s}' \wedge$ (MILP (min, 0, $N(X, U, Y, X') \wedge \Gamma(X) = \hat{s} \wedge \Gamma(U) = \hat{u} \wedge \Gamma(X') = \hat{s}'$) is feasible) **then**
14. $\hat{N} \leftarrow \hat{N} \cup \{(\hat{s}, \hat{u}, \hat{s}')\}$
15. **return** $(\hat{N}, \hat{I}, \hat{G})$

6.2 Control Software With a Guaranteed WCET

From controller $\hat{K}$ computed by QKS (see Sect. 6) we generate our correct-by-construction control software (obdd2c($\hat{K}$)). This is done (function obdd2c) by translating the OBDD representing $\hat{K}$ into C code along the lines of [29]. From such a construction we can easily compute the *Worst Case Execution Time* (WCET) for our controller. We have: $WCET = nrT_B$, where r $[n]$ is the number of bits used to represent plant actions [states] and T_B is the time needed to execute the C instructions modelling the `if-then-else` semantics of OBDD nodes as well as edge complementation (since we use the CUDD package).

Let T be the chosen sampling time. Then it must be: $WCET \leq T$. That is, $nrT_B \leq T$. This equation allows us to know, before hand, the realizability (e.g. with respect to schedulability constraints) of the (to be designed) control software. For example, let $T_B = 10^{-7}$secs, $n = 10$ and $r = 1$. Then, the for the system sampling time we have: $T \geq 10^{-6} = WCET$.

7 Experimental Results

We implemented QKS (Sect. 6) in C, using GLPK to solve MILP problems and the CUDD package for OBDD based computations.

Our experiments aim at evaluating effectiveness of: control abstraction ($\hat{\mathcal{Q}}$, Sect. 6.1) generation, synthesis of OBDD representation of control law ($\hat{K}$, Sect. 6), control software (`obdd2c`($\hat{K}$), Sect. 6.2) size and guaranteed operational ranges (i.e. controllable region). Note that control software reaction time (WCET) is known a priori from Sect. 6.2 and its robustness to parameter variations in the controlled system ($\mathcal{H}$) as well as enforcement of safety bounds on state variables are an input to our synthesis algorithm (e.g. see Ex. 1, 2).

We present experimental results obtained by using QKS on the buck DC-DC converter described in Ex. 2. We denote with $\mathcal{H}$ the DTLHS modeling such a converter. We set the parameters of $\mathcal{H}$ as follows: $T = 10^{-6}$ secs, $L = 2 \cdot 10^{-4}$ H, $r_L = 0.1\ \Omega$, $r_C = 0.1\ \Omega$, $R = 5 \pm 25\%\ \Omega$, $C = 5 \cdot 10^{-5}$ F, $V_i = 15 \pm 25\%$ V and require our controller to be robust to foreseen variations (25%) in the load (R) and in the power supply (V_i).

The model in Ex. 2 already accounts for variations in the power supply. Variations in the load R can be taken into account along the same lines, however much more work is needed (along the lines of [15]) since $\mathcal{H}$ dynamics is not linear in R. This adds 11 auxiliary boolean variables to the model in Ex. 2. Details are in [23]. For converters, *safety* (as well as physical) considerations set requirements on admissible values for state variables. We set: $\mathcal{D}_{i_L} = [-4, 4]$, $\mathcal{D}_{v_O} = [-1, 7]$. Note that robustness requires that, notwithstanding nondeterministic variations (within the given tolerances) for power supply and load, the synthesized controller always keeps state variables within their admissible regions. We use the following bounds for auxiliary variables: $\mathcal{D}_{i_u} = \mathcal{D}_{i_D} = [-10^3, 10^3]$ and $\mathcal{D}_{v_u} = \mathcal{D}_{v_D} = [-10^7, 10^7]$. The initial region I and goal region G are as in Ex. 5. Finally, the DTLHS control problem we consider is $P = (\mathcal{H}, I, G)$. Note that no (formally proved) robust control software is available for buck DC-DC converters.

Table 1. Buck DC-DC converter (Sect. 3): control abstraction and controller synthesis results. Experiments run on an Intel 3.0 GHz Dual Core Linux PC with 4 GB of RAM.

	Control Abstraction				Controller Synthesis		Total
b	CPU	Arcs	MaxLoops	LoopFrac	CPU	OBDD	CPU
8	2.50e+03	1.35e+06	2.54e+04	0.00323	0.00e+00	1.07e+02	2.50e+03
9	1.13e+04	7.72e+06	1.87e+04	0.00440	1.00e+02	1.24e+03	1.14e+04
10	6.94e+04	5.14e+07	2.09e+04	0.00781	7.00e+02	2.75e+03	7.01e+04
11	4.08e+05	4.24e+08	2.29e+04	0.01417	5.00e+03	7.00e+03	4.13e+05

We use a uniform quantization dividing the domain of each state variable (i_L, v_O) into 2^b equal intervals, where b is the number of bits used by AD conversion. We call the resulting quantization Γ_b. The *quantization step* is $\|\Gamma_b\| = 2^{3-b}$.

For each value of interest for b, following Sect. 6, we compute: (1) a (close to minimum) Γ_b control abstraction $\hat{\mathcal{H}}^b$ for $\mathcal{H}$, (2) the mgo strong solution $\hat{K}^b$ for $\hat{\mathcal{P}}^b = (\hat{\mathcal{H}}^b, \varnothing, \Gamma_b(G))$, (3) $\hat{K}^b$ controllable region $\hat{D}^b = \mathrm{Dom}(\hat{K}^b)$, (4) a Γ_b QFC

solution $K^b(s,u) = \hat{K}^b(\Gamma_b(s), \Gamma_b(u))$ to the control problem $\mathcal{P}^b = (\mathcal{H}, \Gamma_b^{-1}(\hat{D}^b), G)$. Note that, since we have two quantized variables (i_L, v_O) each one with b bits, the number of states in the control abstraction is exactly 2^{2b}.

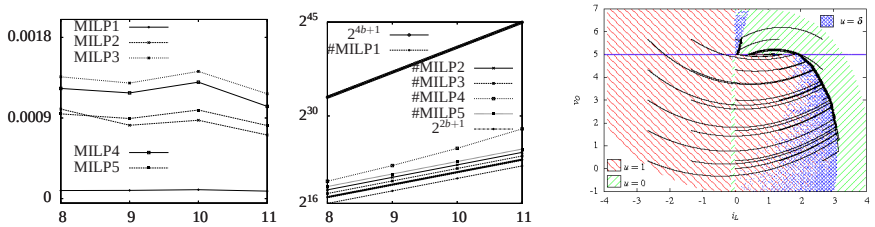
Tab. 1 shows our experimental results. Columns in Tab. 1 have the following meaning. Column b shows the number of AD bits. Columns labelled *Control Abstraction* show performances for Alg. 1. Column *CPU* shows Alg. 1 time (in secs) to compute $\hat{\mathcal{H}}^b$. Column *Arcs* shows the number of transitions in $\hat{\mathcal{H}}^b$. In order to assess effectiveness of function SelfLoop (Sect. 6.1) column *MaxLoops* shows the number of loops in the maximum Γ_b control abstraction for $\mathcal{H}$, while column *LoopFrac* shows the fraction of such loops in $\hat{\mathcal{H}}^b$. Columns labelled *Controller Synthesis* show the computation time in secs (*CPU*) for the generation of $\hat{K}^b$, and the size of its OBDD representation (*OBDD*). The latter is also the size (number of lines) of the C code for our synthesized implementation of $\hat{K}^b$ (obdd2c($\hat{K}^b$)). Finally, column *Total* shows the total computation time in secs (*CPU*) for the whole process (i.e., control abstraction plus controller source code generation). All computations were completed using no more than 200MB. As for the value of μ (see Theor. 1), we have that $\mu =$Unk for $b = 8$, and $\mu =$Sol in all other cases.

From Tab. 1 we see that computing control abstractions (i.e. Alg. 1) is the most expensive operation in QKS (see Sect. 6) and that thanks to function SelfLoop $\hat{K}^b$ contains no more than 2% of the loops in the maximum Γ_b control abstraction for $\mathcal{H}$.

For each MILP problem in Alg. 1, Fig. 3(b) shows (as a function of b) the number of MILP instances solved while Fig. 3(a) shows (as a function of b) the average CPU time (in seconds) spent solving a single MILP problem instance. CPU time standard deviation is always less than 0.003. The correspondence between the curves in Figs. 3(b), 3(a) and Alg. 1 is the following. MILP1 refers to line 3 (and represents also the data for the twin MILP in line 4). MILP2 refers to MILP problems in function SelfLoop (line 7). MILP3 refers to line 9 (and represents also the data for the twin MILP in line 10). MILP4 refers to line 13 and MILP5 refers to line 6.

From Fig.3(a) we see that the average time spent solving each MILP instance is small. The lower [upper] bound to the number of times MILP4 (i.e. the most called MILP in Alg. 1) is called (#MILP4) is $|\Gamma(\mathcal{D}_X)||\Gamma(\mathcal{D}_U)| = 2^{2b+1}$ $[|\Gamma(\mathcal{D}_X)|^2|\Gamma(\mathcal{D}_U)| = 2^{4b+1}]$ (see Remark 3). From Fig. 3(b) we see that #MILP4 is quite close to $|\Gamma(\mathcal{D}_X)||\Gamma(\mathcal{D}_U)| = 2^{2b+1}$. This shows effectiveness of our heuristic to tightly overapproximate Over_Img (lines 9–11 of Alg. 1).

One of the most important features of our approach is that it returns the guaranteed operational range (precondition) of the synthesized software (Theor. 1). This is the *controllable region* D returned by QKS in Sect. 6. Fig. 3(c) shows the controllable region D for K^{10} along with some trajectories (with time increasing counterclockwise) for the closed loop system. Since for $b = 10$ we have $\mu =$ Sol, we have that $I \subseteq D$ (see also Fig. 3(c)). Thus we know (on a formal ground) that 10 bit AD ($\|\Gamma_{10}\| = 2^{-7}$) conversion suffices for our purposes. The controllable region for K^{11} turns out to be only slightly larger than the one for K^{10}.



(a) Average execution time for MILP problems in Alg.1

(b) Number of calls to MILP problems in Alg.1

(c) Controllable region with $b = 10$ bits (δ stands for "don't care")

Fig. 3. QKS performance

8 Conclusion

We presented an effective algorithm that given a DTLHS $\mathcal{H}$ and a quantization schema returns a correct-by-construction robust control software K for $\mathcal{H}$ along with the controllable region R for K. Furthermore, our control software has a WCET linear in the number of bits of the quantization schema. We have implemented our algorithm and shown feasibility of our approach by presenting experimental results on using it to synthesize C controllers for the buck DC-DC converter. Our approach is explicit in the quantized state variables and symbolic in the system modes. Accordingly, it works well with systems with a small number of (continuous) state variables and possibly many modes. Many hybrid systems fall in this category.

Future research may investigate fully symbolic approaches, e.g., based on Fourier-Motzkin (FM) variable elimination, to compute control abstractions. Since FM tools typically work on rational numbers this would also have the effect of avoiding possible numerical errors of MILP solvers [24].

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